Proof by Induction

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1 Mathematical Induction basic proofs

Lets's being with a formula $e^{i\pi} + 1 = 0$ but we can also do

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

Prove that for each integer $n \geq 1$,

$$A(n) : 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Base Case:

Proving that it works for $n = 1$ is the first step in showing the formula works:

For $n = 1$:

$$A(1) = 1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Therefore, the base case is true.

Inductive Hypothesis:

The second phase of the proof requires the assumption that the formula is true for any integer $n \geq 1$:

$$A(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Inductive Step:

The third phase requires that the formula hold for $n+1$:

$$A(n+1) = 1 + 2 + \cdots + n + (n+1)$$

Using the inductive hypothesis, we have:

$$A(n+1) = \frac{n(n+1)}{2} + (n+1)$$

Combine the terms and simplify:

$$A(n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Thus, the formula holds for $n+1$, and by mathematical induction, it holds for all $n \geq 1$

Proposition: If $n \in N$, then

$$1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2.$$

Proof: We will prove this by mathematical induction.

Base Case: For $n = 1$, the left-hand side is simply the first odd number:

$$1 = 1^2.$$

This is true, so the base case holds.

Inductive Hypothesis: Assume that the formula is true for some arbitrary k, i.e., assume that:

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

This is called the *inductive hypothesis.*

Inductive Step: We must show that if the formula is true for k, then it is also true for $k+1$. That is, we need to show:

$$1 + 3 + 5 + \cdots + (2k-1) + [2(k+1) - 1] = (k+1)^2.$$

Using the inductive hypothesis, we know:

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

Adding the next odd number $2(k+1) - 1 = 2k + 1$, we have:

$$k^2 + (2k+1).$$

Simplifying the right-hand side:

$$k^2 + 2k + 1 = (k+1)^2.$$

Thus, the formula holds for $k + 1$.

Conclusion: Since the formula holds for $n = 1$ (the base case), and we have shown that if it holds for $n = k$, it must also hold for $n = k + 1$, by the principle of mathematical induction, the formula is true for all $n \in N$.

Therefore, we have proven that:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

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Proposition: If n is a non-negative integer, then $5 \mid (n^5 - n)$.

Proof: We will prove the proposition by mathematical induction.

Step 1: Base Case ($n = 0$):

We need to check if the proposition holds when $n = 0$.

$$n^5 - n = 0^5 - 0 = 0$$

Since 5 divides 0, the base case holds true.

Step 2: Inductive Hypothesis:

Assume that the proposition holds for some non-negative integer k, i.e.,

$$5 \mid (k^5 - k)$$

This means there exists some integer m such that:

$$k^5 - k = 5m$$

Step 3: Inductive Step:

We need to prove that the proposition holds for $k + 1$, i.e., we need to show that $5 \mid ((k+1)^5 - (k+1))$.

Expand $(k+1)^5$ using the binomial theorem:

$$(k+1)^5 = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

Now subtract $(k + 1)$ from this expression:

$$(k+1)^5 - (k+1) = \left(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\right) - (k+1)$$

Simplifying:

$$(k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 4k$$

Factor out 5 from the terms where it's present:

$$(k+1)^5 - (k+1) = (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$$

By the inductive hypothesis, we know that $5 \mid (k^5 - k)$. Thus, $k^5 - k = 5m$ for some integer m.

Therefore:

$$(k+1)^5 - (k+1) = 5m + 5(k^4 + 2k^3 + 2k^2 + k)$$

$$= 5(m + k^4 + 2k^3 + 2k^2 + k)$$

Since $m + k^4 + 2k^3 + 2k^2 + k$ is an integer, this shows that $5 \mid ((k+1)^5 - (k+1))$.

Conclusion:

By the principle of mathematical induction, we have shown that $5 \mid (n^5 - n)$ for all non-negative integers n. Thus, the proposition is true.